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Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology. Part 2. Stability considerations

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The stability (i) of fully three-dimensional magnetostatic equilibria of arbitrarily complex topology, and (ii) of the analogous steady solutions of the Euler equations of incompressible inviscid flow, are investigated through construction of the second variations $\delta^2 M$ and $\delta^2 K$ of the magnetic energy and kinetic energy with respect to a virtual displacement field $\boldsymbol{\eta}(\boldsymbol{x})$ about the equilibrium configuration. The expressions for $\delta^2 M$ and $\delta^2 K$ differ because in case (i) the magnetic lines of force are frozen in the fluid as it undergoes displacement, whereas in case (ii) the vortex lines are frozen, so that the analogy between magnetic field and *velocity* field on which the existence of steady flows is based does not extend to the perturbed states. It is shown that the stability condition $\delta^2 M > 0$ for all $\boldsymbol{\eta}(\boldsymbol{x})$ for the magnetostatic case can be converted to a form that does not involve the arbitrary displacement $\boldsymbol{\eta}(\boldsymbol{x})$, whereas the condition $\delta^2 K > 0$ for all $\boldsymbol{\eta}$ for the stability of the analogous Euler flow cannot in general be so transformed. Nevertheless it is shown that, if $\delta^2 M$ and $\delta^2 K$ are evaluated for the same basic equilibrium field, then quite generally

$$\delta^2 M + \delta^2 K > 0 \quad (\text{all non-trivial } \boldsymbol{\eta}).$$

A number of special cases are treated in detail. In particular, it is shown that the space-periodic Beltrami field

$$\boldsymbol{B}^{\mathrm{E}} = (B_3 \cos\alpha z + B_2 \sin\alpha y,\ B_1 \cos\alpha x + B_3 \sin\alpha z,\ B_2 \cos\alpha y + B_1 \sin\alpha x)$$

is stable (i.e. $\delta^2 M > 0$ for all $\boldsymbol{\eta}$) and that the medium responds in an elastic manner to perturbations on a scale large compared with α^{-1}. By contrast, it is shown that $\delta^2 K$ is indefinite in sign for the analogous Euler flow, and it is argued that the flow is unstable to certain large-scale helical perturbations having the same sign of helicity as the unperturbed flow. It is conjectured that all topologically non-trivial Euler flows are similarly unstable.

1. Introduction

In a previous paper (Moffatt 1985, hereafter referred to as M85), the existence of magnetostatic equilibria, and hence of analogous steady Euler flows has been established, through consideration of the relaxation of a magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$ in a perfectly conducting, but viscous, fluid contained within a bounded domain $\mathscr{D}$ with fixed boundary $\partial\mathscr{D}$. The magnetic energy stored in an arbitrary initial field $\boldsymbol{B}_0(\boldsymbol{x})$ is converted to kinetic energy of motion and hence, via viscous dissipation, to heat. However, if the topology of the magnetic field is non-trivial, topological constraints place a lower bound on the magnetic energy, and the system asymptotes to a magnetostatic equilibrium $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$ satisfying

$$\left.\begin{aligned} \boldsymbol{j}^{\mathrm{E}} \times \boldsymbol{B}^{\mathrm{E}} = \nabla p^{\mathrm{E}}, \quad \boldsymbol{j}^{\mathrm{E}} = \nabla \times \boldsymbol{B}^{\mathrm{E}}, \quad \nabla \cdot \boldsymbol{B}^{\mathrm{E}} = 0 \quad \text{in } \mathscr{D}, \\ \boldsymbol{n} \cdot \boldsymbol{B}^{\mathrm{E}} = 0 \quad \text{on } \partial\mathscr{D}, \end{aligned}\right\} \tag{1.1}$$

for some scalar (pressure) field $p^{\mathrm{E}}(\boldsymbol{x})$. An essential property of the magnetostatic equilibria satisfying (1.1) is that they may contain tangential discontinuities of $\boldsymbol{B}^{\mathrm{E}}$ (i.e. current sheets) imbedded within $\mathscr{D}$ (as well as on the boundary $\partial\mathscr{D}$), even although the initial field $\boldsymbol{B}_0(\boldsymbol{x})$ is everywhere differentiable (C^1).†

The physical nature of this relaxation process implies that, in general, the equilibrium described by (1.1) should be stable with respect to small displacements of the fluid which conserve the 'frozen-in' character of the magnetic field, since otherwise the decrease in magnetic energy by the above mechanism may (and in general will) continue. Thus we expect that the magnetic energy should be a minimum with respect to small frozen-field perturbations, and this should be true even if the equilibrium field involves current sheets. We are of course ignoring here the possibility of resistive instabilities which may occur as a result of the finite conductivity of real fluids.

To each magnetostatic equilibrium $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$, there corresponds, via the analogy $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x}) \leftrightarrow \boldsymbol{u}^{\mathrm{E}}(\boldsymbol{x})$, a solution of the steady Euler equations

$$\boldsymbol{u}^{\mathrm{E}} \times \boldsymbol{\omega}^{\mathrm{E}} = \nabla h^{\mathrm{E}}, \quad \boldsymbol{\omega} = \nabla \times \boldsymbol{u}^{\mathrm{E}}, \quad \nabla \cdot \boldsymbol{u}^{\mathrm{E}} = 0 \quad \text{in } \mathscr{D}, \quad \boldsymbol{n} \cdot \boldsymbol{u}^{\mathrm{E}} = 0 \quad \text{on } \partial\mathscr{D}, \tag{1.2}$$

describing the steady flow of an inviscid incompressible fluid within $\mathscr{D}$. Note here that $\mathscr{D}$ may be simply or multiply connected, and that $\partial\mathscr{D}$ may consist of several disjoint parts.

As emphasised in M85, the analogy between magnetostatic equilibria and Euler flows exists only for the steady states, and does not extend to questions of stability about these steady states. Thus, the fact that a magnetostatic equilibrium is stable is no guarantee that the corresponding Euler flow is stable. Indeed, if current sheets are present in the former, then vortex sheets are present in the latter, and these are likely to be unstable by the Kelvin–Helmholtz ideal-fluid instability mechanism.

Our aim in this paper is to investigate the relationship, if any, between the two stability problems. We shall show in fact that there is an interesting complementarity between the two problems, with some points of comparison, and also, as expected, some vital differences. The natural tools for this investigation are certain variational techniques, developed in the magnetostatic context by Bernstein *et al.* (1958), and in the Euler flow context by Arnol'd (1966*a*). These techniques differ in the two contexts in a rather subtle manner. We shall therefore develop them *ab initio* here, in order to highlight points of comparison and points of contrast. We shall then illustrate the techniques for three distinct field (or flow) geometries: (i) the two-dimensional situation; (ii) the stability of a cylindrically symmetric field (and of its Euler-flow analogue) to axisymmetric disturbances; and (iii) the stability of the

† It is conceivable, as pointed out by a referee, that the general magnetostatic equilibrium (and therefore the general analogous Euler flow) may have a more complex structure than indicated here; e.g. tangential discontinuities may conceivably be stacked on top of each other in such a way that the field is almost nowhere differentiable. While accepting this possibility, we restrict attention in this paper to fields $\boldsymbol{B}^{\mathrm{E}}$ that have at most a finite number of discontinuities per unit length on any straight line transversal. The current density $\boldsymbol{j}^{\mathrm{E}}$ has a δ-function structure at each such discontinuity, and volume integrals such as (2.21) below must be interpreted in the obvious way when such discontinuities are present.

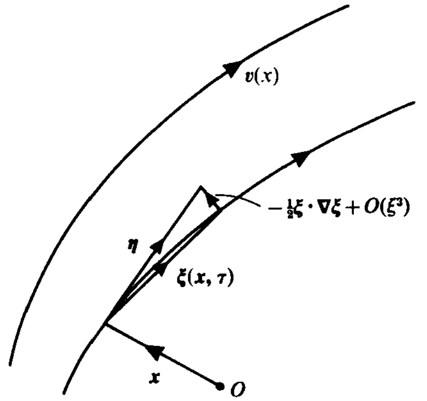

FIGURE 1. Relationship between $\xi(x, \tau)$ and $\eta(x)$ (see (2.5)).

space-periodic Beltrami flows, whose structure has recently been investigated by Dombré *et al.* (1986).

In general, the results are consistent with the conjecture, hinted at in M85, and now made explicit here, that a three-dimensional Euler flow of any significant complexity is in general unstable; in the language of dynamical systems these Euler flows may be regarded as the unstable fixed points in the function space in which solutions of the unsteady Euler equations evolve. Intrinsic instability of complex three-dimensional flow in the inviscid limit is of course one of the hallmarks of turbulence; and it is very much with an eye to application in the turbulence context that this investigation, which might otherwise seem somewhat artificial, is pursued.

2. Stability of magnetostatic equilibria

The magnetic energy associated with a field distribution $\boldsymbol{B}(\boldsymbol{x})$ is

$$M = \frac{1}{2}\int \boldsymbol{B}^2 \,\mathrm{d}V, \tag{2.1}$$

and the equilibrium $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$ satisfying (1.1) will be stable if the corresponding energy M^{E} is minimal with respect to small displacements of the fluid in $\mathscr{D}$ which perturb the field in a 'frozen-field' manner. We restrict attention to incompressible fluids, and suppose that the displacement occurs through the action of a steady solenoidal velocity field $\boldsymbol{v}(\boldsymbol{x})$ ($\nabla\cdot\boldsymbol{v} = 0$, $\boldsymbol{n}\cdot\boldsymbol{v} = 0$ on $\partial\mathscr{D}$) which acts during a small time interval τ.

Let $\boldsymbol{\xi}(\boldsymbol{x}, t)$ be the displacement of the fluid particle initially at $\boldsymbol{x}$; then

$$\frac{\partial \boldsymbol{\xi}}{\partial t} = \boldsymbol{v}(\boldsymbol{x}+\boldsymbol{\xi}) = \boldsymbol{v}(\boldsymbol{x}) + \boldsymbol{\xi}\cdot\nabla\boldsymbol{v} + \dots, \tag{2.2}$$

so that

$$\boldsymbol{\xi}(\boldsymbol{x}, \tau) = \tau\boldsymbol{v}(\boldsymbol{x}) + \tfrac{1}{2}\tau^2\boldsymbol{v}\cdot\nabla\boldsymbol{v} + O(\tau^3). \tag{2.3}$$

Equivalently

$$\boldsymbol{\xi}(\boldsymbol{x}, \tau) = \boldsymbol{\eta}(\boldsymbol{x}) + \tfrac{1}{2}\boldsymbol{\eta}\cdot\nabla\boldsymbol{\eta} + O(\eta^3), \tag{2.4}$$

where $\boldsymbol{\eta}(\boldsymbol{x}) = \tau\boldsymbol{v}(\boldsymbol{x})$. Equation (2.4) may of course be inverted to give

$$\boldsymbol{\eta}(\boldsymbol{x}) = \boldsymbol{\xi} - \tfrac{1}{2}\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi} + O(\xi^3), \tag{2.5}$$

so that $\boldsymbol{\eta}(\boldsymbol{x})$ may be regarded as determined by the displacement field. We shall in fact refer to $\boldsymbol{\eta}(\boldsymbol{x})$ as the 'displacement field' from now on, although strictly this is correct only at linear order. The relationship between $\boldsymbol{\eta}(\boldsymbol{x})$ and $\boldsymbol{\xi}(\boldsymbol{x}, \tau)$ is indicated in figure 1. Note that $\boldsymbol{\eta}$ satisfies

$$\nabla \cdot \boldsymbol{\eta} = 0, \quad \boldsymbol{\eta} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\mathscr{D}. \tag{2.6}$$

A function $\boldsymbol{\eta}(\boldsymbol{x})$, satisfying these conditions will be described as a 'kinematically admissible displacement field' (Arnol'd 1966*a*).

Now the frozen-field equation for $\boldsymbol{B}(\boldsymbol{x}, t)$ is

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v}(\boldsymbol{x}) \times \boldsymbol{B}), \tag{2.7}$$

so that, for $0 < t < \tau$,

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times \{\boldsymbol{v}(\boldsymbol{x}) \times (\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x}) + t\nabla \times (\boldsymbol{v}(\boldsymbol{x}) \times \boldsymbol{B}^{\mathrm{E}}) + \ldots)\}. \tag{2.8}$$

Hence the perturbed field at time τ is

$$\boldsymbol{B}(\boldsymbol{x}, \tau) = \boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x}) + \delta^1\boldsymbol{B} + \delta^2\boldsymbol{B} + O(\eta^3), \tag{2.9}$$

where

$$\delta^1\boldsymbol{B} = \nabla \times (\boldsymbol{\eta} \times \boldsymbol{B}^{\mathrm{E}}), \tag{2.10}$$

$$\delta^2\boldsymbol{B} = \tfrac{1}{2}\nabla \times (\boldsymbol{\eta} \times \delta^1\boldsymbol{B}). \tag{2.11}$$

From (2.1), the perturbed magnetic energy is then

$$M = M^{\mathrm{E}} + \delta^1 M + \delta^2 M + O(\eta^3), \tag{2.12}$$

where

$$\delta^1 M = \int \boldsymbol{B}^{\mathrm{E}} \cdot \delta^1\boldsymbol{B}\,\mathrm{d}V, \tag{2.13}$$

$$\delta^2 M = \tfrac{1}{2}\int [(\delta^1\boldsymbol{B})^2 + 2\boldsymbol{B}^{\mathrm{E}} \cdot \delta^2\boldsymbol{B}]\,\mathrm{d}V. \tag{2.14}$$

All volume integrals are over the domain $\mathscr{D}$, and all surface integrals are over $\partial\mathscr{D}$.

There is one trivial displacement field which we should immediately dispose of, and that is a field of the form

$$\boldsymbol{\eta}(\boldsymbol{x}) = \alpha(\boldsymbol{x})\,\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x}) \quad \text{with } \boldsymbol{B}^{\mathrm{E}} \cdot \nabla\alpha = 0, \tag{2.15}$$

which certainly satisfies (2.6). For this field,

$$\delta^1\boldsymbol{B} = 0, \quad \delta^2\boldsymbol{B} = 0 \quad \text{and so } \delta^2 M = 0 \text{ also.}$$

This displacement along the field lines does not distort the field in any way, and therefore does not change its energy; the field is therefore neutrally stable to such displacements. The same is evidently true for the wider class of displacement fields satisfying

$$\nabla \times (\boldsymbol{\eta} \times \boldsymbol{B}^{\mathrm{E}}) = 0 \quad \text{in } \mathscr{D}. \tag{2.16}$$

We shall describe such displacement fields as trivial with respect to $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$, and in what follows, we restrict attention to the more interesting non-trivial fields for which $\delta^1\boldsymbol{B} \neq 0$.

We expect of course that M must be stationary for small variations about magnetostatic equilibrium, i.e. that $\delta^1 M = 0$. This is easily verified: note first that, using $\boldsymbol{n}\cdot\boldsymbol{B}^{\mathrm{E}} = \boldsymbol{n}\cdot\boldsymbol{\eta} = 0$ on $\partial\mathscr{D}$,

$$\int \nabla\cdot[\boldsymbol{B}^{\mathrm{E}}\times(\boldsymbol{\eta}\times\boldsymbol{B}^{\mathrm{E}})]\,\mathrm{d}V = \int_{\partial\mathscr{D}} \boldsymbol{n}\cdot[\boldsymbol{B}^{\mathrm{E}}\times(\boldsymbol{\eta}\times\boldsymbol{B}^{\mathrm{E}})]\,\mathrm{d}S = 0, \tag{2.17}$$

and

$$\int \boldsymbol{\eta}\cdot\nabla p^{\mathrm{E}}\,\mathrm{d}V = \int_{\partial\mathscr{D}} (\boldsymbol{n}\cdot\boldsymbol{\eta})\,p^{\mathrm{E}}\,\mathrm{d}S = 0. \tag{2.18}$$

Hence

$$\begin{aligned}\delta^1 M = \int \boldsymbol{B}^{\mathrm{E}}\cdot\nabla\times(\boldsymbol{\eta}\times\boldsymbol{B}^{\mathrm{E}})\,\mathrm{d}V &= \int (\nabla\times\boldsymbol{B}^{\mathrm{E}})\cdot(\boldsymbol{\eta}\times\boldsymbol{B}^{\mathrm{E}})\,\mathrm{d}V \\ &= -\int (\boldsymbol{j}^{\mathrm{E}}\times\boldsymbol{B}^{\mathrm{E}})\cdot\boldsymbol{\eta}\,\mathrm{d}V = -\int \boldsymbol{\eta}\cdot\nabla p^{\mathrm{E}}\,\mathrm{d}V = 0.\end{aligned} \tag{2.19}$$

Consider now the expression for $\delta^2 M$:

$$\delta^2 M = \frac{1}{2}\int [(\delta^1\boldsymbol{B})^2 + \boldsymbol{B}^{\mathrm{E}}\cdot\nabla\times(\boldsymbol{\eta}\times\delta^1\boldsymbol{B})]\,\mathrm{d}V, \tag{2.20}$$

so that, integrating the second term by parts.

$$\delta^2 M = \frac{1}{2}\int [(\delta^1\boldsymbol{B})^2 - (\boldsymbol{\eta}\times\boldsymbol{j}^{\mathrm{E}})\cdot\delta^1\boldsymbol{B}]\,\mathrm{d}V. \tag{2.21}$$

The equilibrium $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$ is stable to small disturbances if $\delta^2 M > 0$ for all $\boldsymbol{\eta}(\boldsymbol{x})$ satisfying (2.6). Clearly therefore, in order to find a useful criterion for stability, we need to place a bound on the magnitude of the second term of the integral in (2.21). There are two procedures that we may follow, the first of which is the more straightforward, and the second of which is more naturally related to the corresponding Euler-flow stability problem considered in the next section.

Procedure A

First note that, from the Schwarz inequality,

$$\left|\int (\boldsymbol{\eta}\times\boldsymbol{j}^{\mathrm{E}})\cdot\delta^1\boldsymbol{B}\,\mathrm{d}V\right| \leqslant \left\{\int (\boldsymbol{\eta}\times\boldsymbol{j}^{\mathrm{E}})^2\,\mathrm{d}V \int (\delta^1\boldsymbol{B})^2\,\mathrm{d}V\right\}^{\frac{1}{2}}. \tag{2.22}$$

Secondly, we seek the (non-trivial) displacement $\boldsymbol{\eta}_1(\boldsymbol{x})$ satisfying (2.6) which minimizes the ratio

$$\lambda\{\boldsymbol{\eta}(\boldsymbol{x})\} = \frac{\int (\delta^1\boldsymbol{B})^2\,\mathrm{d}V}{\int (\boldsymbol{\eta}\times\boldsymbol{j}^{\mathrm{E}})^2\,\mathrm{d}V} = \frac{\int [\nabla\times(\boldsymbol{\eta}\times\boldsymbol{B}^{\mathrm{E}})]^2\,\mathrm{d}V}{\int [\boldsymbol{\eta}\times(\nabla\times\boldsymbol{B}^{\mathrm{E}})]^2\,\mathrm{d}V} \tag{2.23}$$

for fixed $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$. To this end, let $\boldsymbol{\eta} = \boldsymbol{\eta}_1(\boldsymbol{x}) + \delta\boldsymbol{\eta}(\boldsymbol{x})$ where $\delta\boldsymbol{\eta}$ is a (virtual) displacement which also satisfies (2.6); then $\boldsymbol{\eta}_1(\boldsymbol{x})$ is determined by the variational statement

$$\delta\int \{[\nabla\times(\boldsymbol{\eta}\times\boldsymbol{B}^{\mathrm{E}})]^2 - \lambda_1[\boldsymbol{\eta}\times(\nabla\times\boldsymbol{B}^{\mathrm{E}})]^2\}\,\mathrm{d}V = 0, \tag{2.24}$$

where λ_1 is a Lagrange multiplier. Hence

$$\int \{\nabla\times(\boldsymbol{\eta}\times\boldsymbol{B}^{\mathrm{E}})\cdot\nabla\times(\delta\boldsymbol{\eta}\times\boldsymbol{B}^{\mathrm{E}}) - \lambda_1\,\boldsymbol{\eta}\times(\nabla\times\boldsymbol{B}^{\mathrm{E}})\cdot\delta\boldsymbol{\eta}\times(\nabla\times\boldsymbol{B}^{\mathrm{E}})\}\,\mathrm{d}V = 0. \tag{2.25}$$

Integrating the first term by parts and using the boundary conditions $\boldsymbol{n}\cdot\boldsymbol{B}^{\mathrm{E}} = \boldsymbol{n}\cdot\boldsymbol{\eta} = 0$ on $\partial\mathscr{D}$ (a procedure that will be used repeatedly in what follows), and rearranging, gives

$$\int \delta\boldsymbol{\eta}\cdot\{\boldsymbol{B}^{\mathrm{E}}\times[\nabla\times\nabla\times(\boldsymbol{\eta}\times\boldsymbol{B}^{\mathrm{E}})]-\lambda_1(\nabla\times\boldsymbol{B}^{\mathrm{E}})\times[\boldsymbol{\eta}\times(\nabla\times\boldsymbol{B}^{\mathrm{E}})]\}\,\mathrm{d}V = 0. \tag{2.26}$$

Since this holds for arbitrary solenoidal $\delta\boldsymbol{\eta}$, it follows that the minimizing $\boldsymbol{\eta} = \boldsymbol{\eta}_1(\boldsymbol{x})$ satisfies

$$\boldsymbol{B}^{\mathrm{E}}\times[\nabla\times\nabla\times(\boldsymbol{B}^{\mathrm{E}}\times\boldsymbol{\eta}_1)]+\lambda_1(\nabla\times\boldsymbol{B}^{\mathrm{E}})\times[\boldsymbol{\eta}_1\times(\nabla\times\boldsymbol{B}^{\mathrm{E}})] = \nabla\Psi_1 \tag{2.27}$$

for some scalar field $\Psi_1(\boldsymbol{x})$. Moreover, when this equation is satisfied, together with the boundary condition $\boldsymbol{\eta}_1\cdot\boldsymbol{n} = 0$ on $\partial\mathscr{D}$, scalar multiplication of (2.27) by $\boldsymbol{\eta}_1$ and integration over $\mathscr{D}$ shows that

$$\lambda\{\boldsymbol{\eta}_1(\boldsymbol{x})\} = \lambda_1, \tag{2.28}$$

so that λ_1 is real and positive, and is determined uniquely by $\eta_1(\boldsymbol{x})$.

It now follows from (2.21)–(2.23) and (2.28) that

$$\delta^2 M \geqslant \tfrac{1}{2}(1-\lambda_1^{-\frac{1}{2}})\int(\delta^1\boldsymbol{B})^2\,\mathrm{d}V. \tag{2.29}$$

Hence if λ_1 is the smallest strictly positive eigenvalue of (2.27), then $\lambda_1 > 1$ implies that $\delta^2 M > 0$ for all non-trivial displacement fields $\boldsymbol{\eta}(\boldsymbol{x})$, i.e. that the magnetostatic equilibrium is stable.

Procedure B

Instead of $\boldsymbol{\eta}\times\boldsymbol{j}^{\mathrm{E}}$ in (2.21), we may use the solenoidal part of $\boldsymbol{\eta}\times\boldsymbol{j}^{\mathrm{E}}$ defined by

$$(\boldsymbol{\eta}\times\boldsymbol{j}^{\mathrm{E}})_{\mathrm{s}} = \boldsymbol{\eta}\times\boldsymbol{j}^{\mathrm{E}}+\nabla\phi, \tag{2.30}$$

where ϕ is chosen so that

$$\nabla\cdot(\boldsymbol{\eta}\times\boldsymbol{j}^{\mathrm{E}})_{\mathrm{s}} = 0, \quad \boldsymbol{n}\cdot(\boldsymbol{\eta}\times\boldsymbol{j}^{\mathrm{E}})_{\mathrm{s}} = 0 \quad \text{on } \partial\mathscr{D}. \tag{2.31}$$

These conditions imply that $\phi(\boldsymbol{x})$ ($= \phi\{\boldsymbol{\eta}(\boldsymbol{x})\}$ is the linear functional of $\boldsymbol{\eta}$ (unique up to an additive constant) determined by the Neumann problem

$$\nabla^2\phi = -\nabla\cdot(\boldsymbol{\eta}\times\boldsymbol{j}^{\mathrm{E}}), \quad \frac{\partial\phi}{\partial n} = -\boldsymbol{n}\cdot(\boldsymbol{\eta}\times\boldsymbol{j}^{\mathrm{E}}) \quad \text{on } \partial\mathscr{D}. \tag{2.32}$$

Since

$$\int\nabla\phi\cdot\delta^1\boldsymbol{B}\,\mathrm{d}V = \int\nabla\cdot(\phi\delta^1\boldsymbol{B})\,\mathrm{d}V = \int\phi\boldsymbol{\eta}\cdot\delta^1\boldsymbol{B}\,\mathrm{d}S = 0, \tag{2.33}$$

it follows from (2.21) that

$$\delta^2 M = \frac{1}{2}\int\{[\nabla\times(\boldsymbol{\eta}\times\boldsymbol{B}^{\mathrm{E}})]^2-(\boldsymbol{\eta}\times\boldsymbol{j}^{\mathrm{E}})_{\mathrm{s}}\cdot\nabla\times(\boldsymbol{\eta}\times\boldsymbol{B}^{\mathrm{E}})\}\,\mathrm{d}V. \tag{2.34}$$

Now we may put a bound on the second term as in Procedure A. First

$$\left|\int(\boldsymbol{\eta}\times\boldsymbol{j}^{\mathrm{E}})_{\mathrm{s}}\cdot\nabla\times(\boldsymbol{\eta}\times\boldsymbol{B}^{\mathrm{E}})\,\mathrm{d}V\right| \leqslant \left\{\int(\boldsymbol{\eta}\times\boldsymbol{j}^{\mathrm{E}})_{\mathrm{s}}^2\,\mathrm{d}V\int[\nabla\times(\boldsymbol{\eta}\times\boldsymbol{B}^{\mathrm{E}})]^2\,\mathrm{d}V\right\}^{\frac{1}{2}}, \tag{2.35}$$

and secondly, we seek the non-trivial displacement $\boldsymbol{\eta}_2(\boldsymbol{x})$ which minimizes the (new) ratio

$$\lambda = \frac{\int[\nabla\times(\boldsymbol{\eta}\times\boldsymbol{B}^{\mathrm{E}})]^2\,\mathrm{d}V}{\int(\boldsymbol{\eta}\times\boldsymbol{j}^{\mathrm{E}})_{\mathrm{s}}^2\,\mathrm{d}V}. \tag{2.36}$$

The corresponding variational problem is now

$$\delta\int\{[\nabla\times(\boldsymbol{\eta}\times\boldsymbol{B}^{\mathrm{E}})]^2-\lambda_2(\boldsymbol{\eta}\times\boldsymbol{j}^{\mathrm{E}})_{\mathrm{s}}^2\}\,\mathrm{d}V=0, \tag{2.37}$$

or, using (2.30),

$$\int\{[\nabla\times(\boldsymbol{\eta}\times\boldsymbol{B}^{\mathrm{E}})]\cdot\nabla\times(\delta\boldsymbol{\eta}\times\boldsymbol{B}^{\mathrm{E}})-\lambda_2(\boldsymbol{\eta}\times\boldsymbol{j}^{\mathrm{E}})_{\mathrm{s}}\cdot(\delta\boldsymbol{\eta}\times\boldsymbol{j}^{\mathrm{E}}+\nabla\delta\phi)\}\,\mathrm{d}V=0, \tag{2.38}$$

where $\delta\phi=\phi\{\delta\boldsymbol{\eta}(\boldsymbol{x})\}$. Note now (cf. 2.33) that

$$\int[(\boldsymbol{\eta}\times\boldsymbol{j}^{\mathrm{E}})_{\mathrm{s}}\cdot\nabla\delta\phi]\,\mathrm{d}V=0, \tag{2.39}$$

so that (2.38) may be manipulated to the form

$$\int\delta\boldsymbol{\eta}\cdot\{\boldsymbol{B}^{\mathrm{E}}\times[\nabla\times\nabla\times(\boldsymbol{\eta}\times\boldsymbol{B}^{\mathrm{E}})]-\lambda_2\boldsymbol{j}^{\mathrm{E}}\times(\boldsymbol{\eta}\times\boldsymbol{j}^{\mathrm{E}}+\nabla\phi)\}\,\mathrm{d}V=0. \tag{2.40}$$

Hence, in this case, the minimizing $\boldsymbol{\eta}=\boldsymbol{\eta}_2(\boldsymbol{x})$ satisfies

$$\boldsymbol{B}^{\mathrm{E}}\times[\nabla\times\nabla\times(\boldsymbol{B}^{\mathrm{E}}\times\boldsymbol{\eta}_2)]+\lambda_2\boldsymbol{j}^{\mathrm{E}}\times(\boldsymbol{\eta}\times\boldsymbol{j}^{\mathrm{E}}+\nabla\phi_2)=\nabla\Psi_2 \tag{2.41}$$

for some scalar field Ψ_2. Again it is easily verified that

$$\lambda\{\boldsymbol{\eta}_2(\boldsymbol{x})\}=\lambda_2, \tag{2.42}$$

so that λ_2 is real and positive; moreover, now

$$\delta^2M\geqslant\tfrac{1}{2}(1-\lambda_2^{-\frac{1}{2}})\int(\delta^1\boldsymbol{B})^2\,\mathrm{d}V, \tag{2.43}$$

so that if λ_2 is the smallest strictly positive eigenvalue of (2.41) then $\lambda_2>1$ is sufficient for stability of the magnetostatic equilibrium to small disturbances.

Note that, by virtue of (2.39) with $\delta\phi$ replaced by ϕ,

$$\int(\boldsymbol{\eta}\times\boldsymbol{j}^{\mathrm{E}})_{\mathrm{s}}^2\,\mathrm{d}V=\int[(\boldsymbol{\eta}\times\boldsymbol{j})^2-(\nabla\phi)^2]\,\mathrm{d}V. \tag{2.44}$$

Hence

$$\lambda_2=\frac{\int[\nabla\times(\boldsymbol{\eta}_2\times\boldsymbol{B}^{\mathrm{E}})]^2\,\mathrm{d}V}{\int[(\boldsymbol{\eta}_2\times\boldsymbol{j})^2-(\nabla\phi_2)^2]\,\mathrm{d}V}\geqslant\frac{\int[\nabla\times(\boldsymbol{\eta}_2\times\boldsymbol{B}^{\mathrm{E}})]^2\,\mathrm{d}V}{\int(\boldsymbol{\eta}_2\times\boldsymbol{j}^{\mathrm{E}})^2\,\mathrm{d}V}\geqslant\lambda_1, \tag{2.45}$$

with equality only when $\boldsymbol{\eta}_2=\boldsymbol{\eta}_1$ and $\nabla\phi_2=0$. Hence the condition $\lambda_1>1$ for stability is stronger than the condition $\lambda_2>1$. However, it is the form (2.34) that will be more useful for comparison with the Euler-flow stability problem, to which we now turn.

2.1. *The case of force-free fields*

The particular case in which the magnetic field $\boldsymbol{B}^{\mathrm{E}}$ satisfies the 'force-free' condition

$$\nabla\times\boldsymbol{B}^{\mathrm{E}}=\alpha\boldsymbol{B}^{\mathrm{E}} \tag{2.46}$$

has been studied by Molodensky (1974), who showed that, if the disturbance is confined to a region in the interior of $\mathscr{D}$ of maximum diameter L then, in the notation of the present paper,

$$\delta^2M\geqslant\tfrac{1}{2}(1-|\alpha|L)\int_{\mathscr{D}}(\delta^1\boldsymbol{B})^2\,\mathrm{d}V \tag{2.47}$$

(a result that may be deduced from (2.21)), so that the field is stable provided

$$L < |\alpha|^{-1}. \tag{2.48}$$

The case when the disturbance extends to the boundary $\partial\mathscr{D}$ is not covered by Molodensky's analysis. In two further papers, however, Molodensky (1975, 1976) examined the particular case of a spherical domain, and generalized the inequality (2.47) to cover the case when α is a function of position satisfying $\boldsymbol{B}^{\mathrm{E}}\cdot\nabla\alpha = 0$.

A particular force-free field is treated in §6 of the present paper, and is shown to be stable to *all* disturbances, irrespective of lengthscale.

3. Stability of analogous Euler flow

Consider now the analogous Euler flow $\boldsymbol{u}^{\mathrm{E}}(\boldsymbol{x})$ satisfying (1.2). As shown by Arnol'd (1966*a*) the stability of this flow to perturbations governed by the Euler equations of inviscid flow may be investigated through consideration of the kinetic-energy invariant

$$K = \frac{1}{2}\int \boldsymbol{u}^2\,\mathrm{d}V. \tag{3.1}$$

Under perturbations of the flow $\boldsymbol{u}^{\mathrm{E}}(\boldsymbol{x})$, the vortex lines are frozen in the fluid (not the streamlines as would be required if the $\boldsymbol{u}\leftrightarrow\boldsymbol{B}$ analogy were to persist for the stability problem), and the energy K is stationary for $\boldsymbol{u} = \boldsymbol{u}^{\mathrm{E}}$ with respect to such perturbations (see below). The question of stability leads naturally to consideration of the second variation $\delta^2 K$ of the energy with respect to the initial displacement field $\boldsymbol{\eta}(\boldsymbol{x})$. If K is a minimum† with respect to all admissible perturbations (i.e. if $\delta^2 K$ is a positive-definite functional of $\boldsymbol{\eta}$) then $\boldsymbol{u}$ remains in a 'neighbourhood' of $\boldsymbol{u}^{\mathrm{E}}$ in the sense that the norm

$$\|\boldsymbol{u}-\boldsymbol{u}^{\mathrm{E}}\| = |K-K^{\mathrm{E}}|^{\frac{1}{2}} \approx |\delta^2 K|^{\frac{1}{2}} \tag{3.2}$$

remains constant, and the flow is then (in this sense) stable.

We consider then a displacement field $\boldsymbol{\eta}(\boldsymbol{x})$ as in §2, satisfying (2.6). Regarding this as a virtual instantaneous displacement which carries the vortex lines of the equilibrium field $\boldsymbol{\omega}^{\mathrm{E}}$ in a 'frozen-in' manner, the perturbed vorticity field, by analogy with (2.9), is given by

$$\boldsymbol{\omega} = \boldsymbol{\omega}^{\mathrm{E}}(\boldsymbol{x}) + \delta^1\boldsymbol{\omega} + \delta^2\boldsymbol{\omega} + O(\eta^3), \tag{3.3}$$

where

$$\delta^1\boldsymbol{\omega} = \nabla\times(\boldsymbol{\eta}\times\boldsymbol{\omega}^{\mathrm{E}}), \tag{3.4}$$

$$\delta^2\boldsymbol{\omega} = \tfrac{1}{2}\nabla\times(\boldsymbol{\eta}\times\delta^1\boldsymbol{\omega}). \tag{3.5}$$

Again we suppose that the displacement is non-trivial in the sense that $\delta^1\boldsymbol{\omega} \neq 0$. By uncurling (3.3), the corresponding perturbed velocity field is

$$\boldsymbol{u} = \boldsymbol{u}^{\mathrm{E}}(\boldsymbol{x}) + \delta^1\boldsymbol{u} + \delta^2\boldsymbol{u} + O(\eta^3), \tag{3.6}$$

where

$$\delta^1\boldsymbol{u} = (\boldsymbol{\eta}\times\boldsymbol{\omega}^{\mathrm{E}})_{\mathrm{s}}, \tag{3.7}$$

$$\delta^2\boldsymbol{u} = \tfrac{1}{2}(\boldsymbol{\eta}\times\delta^1\boldsymbol{\omega})_{\mathrm{s}}, \tag{3.8}$$

where the suffix s denotes 'solenoidal part of' defined as in (2.30)–(2.32).

† Arnol'd (1966*a*) argued that stability is ensured if K is either a minimum *or* a maximum with respect to small perturbations; but in a subsequent paper (Arnol'd 1966*b*, see also McIntyre & Shepherd 1985) he showed that in the two-dimensional case, maximality of K is not in fact sufficient for stability. We discuss this point in the context of cylindrically symmetric flow in §5 of this paper.

The corresponding expansion for K is then

$$K = K^{\mathrm{E}} + \delta^1 K + \delta^2 K + O(\eta^3), \tag{3.9}$$

where

$$\delta^1 K = \int \boldsymbol{u}^{\mathrm{E}} \cdot \delta^1 \boldsymbol{u} \, \mathrm{d}V, \tag{3.10}$$

$$\delta^2 K = \frac{1}{2} \int [(\delta^1 \boldsymbol{u})^2 + 2\boldsymbol{u}^{\mathrm{E}} \cdot \delta^2 \boldsymbol{u}] \, \mathrm{d}V. \tag{3.11}$$

These expressions were obtained by Arnol'd (1966*a*). It is easily shown that, by virtue of (1.2),

$$\delta^1 K = 0, \tag{3.12}$$

i.e. the kinetic energy is indeed stationary with respect to admissible variations about the equilibrium state $\boldsymbol{u} = \boldsymbol{u}^{\mathrm{E}}$. This is theorem 1 of Arnol'd (1966*a*).

Consider now the second variation $\delta^2 K$. This may be readily manipulated to the form

$$\delta^2 K = \frac{1}{2} \int [(\boldsymbol{\eta} \times \boldsymbol{\omega}^{\mathrm{E}})_{\mathrm{s}}^2 - (\boldsymbol{\eta} \times \boldsymbol{\omega}^{\mathrm{E}})_{\mathrm{s}} \cdot \boldsymbol{\nabla} \times (\boldsymbol{\eta} \times \boldsymbol{u}^{\mathrm{E}})] \, \mathrm{d}V \tag{3.13}$$

using $\boldsymbol{\eta} \cdot \boldsymbol{n} = 0$ on $\partial \mathscr{D}$. Remarkably, the second term in the integral is the exact analogue of the second term in (2.34), under the analogy $\boldsymbol{u}^{\mathrm{E}} \leftrightarrow \boldsymbol{B}^{\mathrm{E}}$, $\boldsymbol{\omega}^{\mathrm{E}} \leftrightarrow \boldsymbol{j}^{\mathrm{E}}$. The first term is, however, different from the first term in (2.34), and it is this difference which permits instability of the Euler flow, despite the fact that the analogous magnetostatic equilibrium is stable.

If we make the substitutions

$$\boldsymbol{B}^{\mathrm{E}} \rightarrow \boldsymbol{u}^{\mathrm{E}}, \quad \boldsymbol{j}^{\mathrm{E}} \rightarrow \boldsymbol{\omega}^{\mathrm{E}} \tag{3.14}$$

in (2.34), we obtain

$$\delta^2 M = \frac{1}{2} \int \{[\boldsymbol{\nabla} \times (\boldsymbol{\eta} \times \boldsymbol{u}^{\mathrm{E}})]^2 - (\boldsymbol{\eta} \times \boldsymbol{\omega}^{\mathrm{E}})_{\mathrm{s}} \cdot \boldsymbol{\nabla} \times (\boldsymbol{\eta} \times \boldsymbol{u}^{\mathrm{E}})\} \, \mathrm{d}V \tag{3.15}$$

and of course, if the analogous magnetostatic equilibrium is stable, then $\delta^2 M \geqslant 0$ for all $\boldsymbol{\eta}$. From (3.13) and (3.15), we have

$$\delta^2 K + \delta^2 M = \frac{1}{2} \int [\boldsymbol{\nabla} \times (\boldsymbol{\eta} \times \boldsymbol{u}^{\mathrm{E}}) - (\boldsymbol{\eta} \times \boldsymbol{\omega}^{\mathrm{E}})_{\mathrm{s}}]^2 \, \mathrm{d}V > 0, \tag{3.16}$$

a result that is of some interest, although unfortunately it does not give sufficient information about $\delta^2 K$ to discriminate between stable and unstable states.

If we apply the argument of Procedure B in §2 to place an upper bound on the second term of (3.13), we find (contrast (2.43)) that

$$\delta^2 K \geqslant \frac{1}{2} \int \{(\boldsymbol{\eta} \times \boldsymbol{\omega}^{\mathrm{E}})_{\mathrm{s}}^2 - \lambda_2^{-\frac{1}{2}} [\boldsymbol{\nabla} \times (\boldsymbol{\eta} \times \boldsymbol{u}^{\mathrm{E}})]^2\} \, \mathrm{d}V, \tag{3.17}$$

where λ_2 is still the smallest eigenvalue of (2.41). Again the statement (3.17) is not particularly helpful because it appears to be impossible to convert the integrand to a form that is positive definite except for certain very simple choices of $\boldsymbol{u}^{\mathrm{E}}(\boldsymbol{x})$. This difficulty (which was recognized by Arnol'd 1966*a*) is perhaps no more than a first indication that, in general, three-dimensional Euler flows of any complexity have an associated $\delta^2 K$ that is indefinite as regards sign, so that the invariance of K places no constraint on the growth of perturbations to the flow. We shall evaluate $\delta^2 K$ explicitly for certain basic flows in the following sections in order to test this assertion.

4. Two-dimensional situations

The case of two-dimensional Euler flow was treated by Arnol'd (1965*a*, 1966*a*, *b*, *c*), and has recently been further explored by McIntyre & Shepherd (1985, and references given therein). We shall focus attention here on the difference between the magnetostatic problem and the Euler-flow problems, as reflected in the expressions (2.21) and (3.13) for $\delta^2 M$ and $\delta^2 K$.

Consider first the magnetostatic situation, with a two-dimensional field of the form

$$\boldsymbol{B}^{\mathrm{E}} = \left(\frac{\partial A}{\partial y}, -\frac{\partial A}{\partial x}, 0\right), \quad A = A(x, y), \tag{4.1}$$

so that

$$\boldsymbol{j}^{\mathrm{E}} = \boldsymbol{\nabla} \times \boldsymbol{B}^{\mathrm{E}} = (0, 0, -\nabla^2 A). \tag{4.2}$$

The domain $\mathscr{D}$ is the cylinder $\mathscr{D}_2 \times \{z: 0 < z < 1\}$, where $\mathscr{D}_2$ is a domain of the (x, y)-plane. Suppose that the boundary $\partial\mathscr{D}_2$ consists of an exterior closed curve C_0 and (possibly) interior closed curves $C_1, C_2, \ldots$. Then the boundary condition on $\boldsymbol{B}^{\mathrm{E}}$ becomes

$$A = A_i \quad \text{on } C_i \quad (i = 0, 1, 2, \ldots), \tag{4.3}$$

where each A_i is constant. Note that the A_i remain constant under frozen-field displacements, since the flux of $\boldsymbol{B}$ across any curve joining C_i to C_0 remains constant. The basic equilibrium condition (1.1) is satisfied provided

$$\nabla^2 A = f(A) \tag{4.4}$$

for some function $f(A)$.

We consider a two-dimensional displacement field

$$\boldsymbol{\eta} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0\right), \tag{4.5}$$

where

$$\psi = \psi_i \quad \text{on } C_i \quad (i = 0, 1, 2, \ldots) \tag{4.6}$$

with each ψ_i constant. In order to evaluate $\delta^2 M$, we first write (2.21) in the equivalent form

$$\delta^2 M = \frac{1}{2}\int [(\delta^1 \boldsymbol{B})^2 - (\boldsymbol{\eta} \times \boldsymbol{B}^{\mathrm{E}}) \cdot \boldsymbol{\nabla} \times (\boldsymbol{\eta} \times \boldsymbol{j}^{\mathrm{E}})] \, \mathrm{d}V. \tag{4.7}$$

Now

$$\boldsymbol{\nabla} \times (\boldsymbol{\eta} \times \boldsymbol{j}^{\mathrm{E}}) = -(\boldsymbol{\eta} \cdot \boldsymbol{\nabla})\boldsymbol{j}^{\mathrm{E}} = -(\boldsymbol{\eta} \times \boldsymbol{B}^{\mathrm{E}}) f'(A), \tag{4.8}$$

so that

$$(\boldsymbol{\eta} \times \boldsymbol{B}^{\mathrm{E}}) \cdot \boldsymbol{\nabla} \times (\boldsymbol{\eta} \times \boldsymbol{j}^{\mathrm{E}}) = -(\boldsymbol{\eta} \times \boldsymbol{B}^{\mathrm{E}})^2 f'(A). \tag{4.9}$$

Moreover, writing

$$\boldsymbol{\eta} \times \boldsymbol{B} = (0, 0, G(x, y)), \tag{4.10}$$

where, obviously, $G = 0$ on $\partial\mathscr{D}_2$, we have

$$\int [\boldsymbol{\nabla} \times (\boldsymbol{\eta} \times \boldsymbol{B})]^2 \, \mathrm{d}V = \int (\boldsymbol{\nabla} G)^2 \, \mathrm{d}x \, \mathrm{d}y \geqslant q_0^2 \int G^2 \, \mathrm{d}x \, \mathrm{d}y, \tag{4.11}$$

where q_0^2 is the smallest eigenvalue of the problem

$$(\nabla^2 + q^2) G = 0, \quad G = 0 \quad \text{on } \partial\mathscr{D}_2. \tag{4.12}$$

Hence from (4.7),

$$\delta^2 M \geqslant \frac{1}{2}\int_{\mathscr{D}_2} (q_0^2 + f'(A)) (\boldsymbol{\eta} \times \boldsymbol{B})^2 \, \mathrm{d}x \, \mathrm{d}y, \tag{4.13}$$

and so a sufficient condition for the equilibrium to be stable to two-dimensional disturbances is

$$f'(A) > -q_0^2 \quad \text{throughout } \mathscr{D}_2. \tag{4.14}$$

Consider now the analogous Euler flow, for which

$$\boldsymbol{u}^{\mathrm{E}} = \left(\frac{\partial A}{\partial y}, -\frac{\partial A}{\partial x}, 0\right), \tag{4.15}$$

where A satisfies (4.3) and (4.4). Hence from (3.13),

$$\delta^2 K = \frac{1}{2}\int_{\mathscr{D}_2} [(\boldsymbol{\eta} \times \boldsymbol{\omega}^{\mathrm{E}})_{\mathrm{s}}^2 + (\boldsymbol{\eta} \times \boldsymbol{u}^{\mathrm{E}})^2 f'(A)]\,\mathrm{d}x\,\mathrm{d}y, \tag{4.16}$$

and so a sufficient condition for stability is

$$f'(A) > 0 \quad \text{throughout } \mathscr{D}_2. \tag{4.17}$$

This is theorem VII of Arnol'd (1966*a*). Clearly the condition (4.17) implies not only stability of the Euler flow, but also (since (4.14) is then also satisfied) stability of the analogous magnetostatic equilibrium.

Note that the second term of the integrand in (4.16) involves only $\boldsymbol{\eta}$ and not the gradient of $\boldsymbol{\eta}$; this is a special feature of the two-dimensional case which permits the extraction of a sufficient condition for stability to all two-dimensional disturbances. In the three-dimensional case (for which $\boldsymbol{\omega}^{\mathrm{E}} \cdot \nabla \boldsymbol{\eta} \neq 0$) this simplification is absent.

5. Cylindrically symmetric situations

A second situation in which the comparison between the magnetostatic equilibrium and the analogous Euler flow is instructive is that in which

$$\boldsymbol{B}^{\mathrm{E}} = (0, b(r), 0), \tag{5.1}$$

or analogously

$$\boldsymbol{u}^{\mathrm{E}} = (0, v(r), 0), \tag{5.2}$$

in cylindrical polar coordinates (r, θ, z). We take $\mathscr{D}$ to be the domain $\{r < r < r_2, 0 < z < z_0\}$. Consider the stability of these states to axisymmetric disturbances, represented by the disturbance field

$$\boldsymbol{\eta} = (\eta_r, 0, \eta_z) = \left(\frac{1}{r}\frac{\partial \psi}{\partial z}, 0, -\frac{1}{r}\frac{\partial \psi}{\partial r}\right). \tag{5.3}$$

Calculation of $\delta^2 M$ and $\delta^2 K$ is straightforward, and we simply state the results:

$$\delta^2 M = -\pi \iint \eta_r^2 \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{b}{r}\right)^2 r^2\,\mathrm{d}r\,\mathrm{d}z, \tag{5.4}$$

and

$$\delta^2 K = 2\pi \iint \eta_r^2 \left[\frac{\mathrm{d}}{\mathrm{d}r}(rv)^2\right] r^{-2}\,\mathrm{d}r\,\mathrm{d}z. \tag{5.5}$$

Hence the magnetostatic equilibrium is stable to disturbances of the form (5.3) if

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{b}{r}\right)^2 < 0, \tag{5.6}$$

a condition that may be obtained in an elementary manner by considering the change in magnetic energy associated with interchange of two flux tubes.

As regards the Euler flow (5.2), note that if

$$\frac{\mathrm{d}}{\mathrm{d}r}(rv)^2 > 0 \tag{5.7}$$

then $\delta^2 K > 0$ for all axisymmetric disturbances, and so the flow is stable to such disturbances. This is the well-known Rayleigh stability criterion.

As mentioned in the footnote in §3. Arnol'd's (1966*a*) original assertion was that stability is assured if *either* $\delta^2 K > 0$ *or* $\delta^2 K < 0$ for all admissible perturbations. The example considered here shows that the second alternative cannot be generally correct. The reason for the failure of the original Arnol'd argument in the case when K is *maximal* for $\boldsymbol{u} = \boldsymbol{u}^{\mathrm{E}}$ is that the energy released in perturbing the flow to an adjacent state is then available to augment the disturbance and so to perturb the flow still further – and so on. This cannot happen when K is minimal for $\boldsymbol{u} = \boldsymbol{u}^{\mathrm{E}}$, and the condition that $\delta^2 K$ is positive-definite *is* therefore a correct sufficient condition for stability.†

Note that, if $b(r)$ is replaced by $v(r)$ in (5.4), then from (5.4) and (5.5).

$$\delta^2 M + \delta^2 E = 4\pi \iint_{\mathscr{D}} \eta_r^2 \left(\frac{v}{r}\right)^2 r\,\mathrm{d}r\,\mathrm{d}z > 0, \tag{5.8}$$

consistent with (3.16).

6. Stability of Beltrami fields

A particular, fully three-dimensional, magnetostatic equilibrium is provided by the force-free (or Beltrami) field

$$\boldsymbol{B}^{\mathrm{E}} = (B_3 \cos\alpha z + B_2 \sin\alpha y,\ B_1 \cos\alpha x + B_3 \sin\alpha z,\ B_2 \cos\alpha y + B_1 \sin\alpha x), \tag{6.1}$$

for which

$$\boldsymbol{j}^{\mathrm{E}} = \nabla \times \boldsymbol{B}^{\mathrm{E}} = -\alpha \boldsymbol{B}^{\mathrm{E}}. \tag{6.2}$$

The topological structure of this field has recently been explored by Dombré *et al.* (1986): provided $B_1 B_2 B_3 \neq 0$, the lines of force are space-filling in a subdomain $\mathscr{D}_c$ of $\mathbb{R}^3$ which is connected from one 'periodicity box' of the field (6.1) to its neighbouring boxes, like a three-dimensional web. In this section we address the question of whether the magnetostatic equilibrium (6.1) is stable; in the following section, we consider the same question for the analogous Euler flow.

It is known (Woltjer 1958) that minimization of magnetic energy subject only to the single constraint of conservation of global magnetic helicity leads to a force-free field satisfying (6.2) for some constant α. However, it is not the case that every such force-free field is the result of such a minimization procedure; and in order to test the stability of a given force-free field such as (6.1), we need to evaluate the integral (2.21) for $\delta^2 M$, to see whether or not it is positive-definite. The question has been considered by Arnol'd (1974) who concluded on general grounds that the field (6.1) is indeed one of minimum energy. We shall here provide explicit verification of this result.

† Dr David Andrews (private communication) has shown that if the procedure of Arnol'd (1965*a*) is followed for the axisymmetric flow considered here, then the condition (5.7) is again obtained as the condition that a certain function $\delta^2 J$ be positive definite, but that if $\mathrm{d}(rv)^2/\mathrm{d}r < 0$, then $\delta^2 J$ is not in general negative definite. This conclusion is entirely consistent with the above discussion.

First we write (6.1) in the more compact form

$$\boldsymbol{B}^{\mathrm{E}} = \sum_n \boldsymbol{B}_n \, \mathrm{e}^{\mathrm{i}\boldsymbol{a}_n \cdot \boldsymbol{x}}, \tag{6.3}$$

where n takes the values ± 1, ± 2, ± 3, and

$$\boldsymbol{B}_{-n} = \boldsymbol{B}_n^*, \quad \boldsymbol{a}_{-n} = -\boldsymbol{a}_n, \quad |\boldsymbol{a}_n| = |\alpha|. \tag{6.4}$$

The general disturbance $\boldsymbol{\eta}$ admits a Fourier representation

$$\boldsymbol{\eta} = \sum_m \boldsymbol{\eta}_m \, \mathrm{e}^{\mathrm{i}\boldsymbol{k}_m \cdot \boldsymbol{x}}, \tag{6.5}$$

where

$$\boldsymbol{\eta}_{-m} = \boldsymbol{\eta}_m^*, \quad \boldsymbol{k}_{-m} = -\boldsymbol{k}_m. \tag{6.6}$$

Now, using (6.2), and replacing the volume integral in (2.21) by a space-average, denoted $\langle \ldots \rangle$, we have

$$\delta^2 M = \langle (\nabla \times \boldsymbol{A}) \cdot (\nabla \times \boldsymbol{A} + \alpha \boldsymbol{A}) \rangle, \tag{6.7}$$

where

$$\boldsymbol{A} = \boldsymbol{\eta} \times \boldsymbol{B}^{\mathrm{E}} = \sum_\lambda \boldsymbol{A}_\lambda \, \mathrm{e}^{\mathrm{i}\boldsymbol{\kappa}_\lambda \cdot \boldsymbol{x}}, \tag{6.8}$$

where λ denotes the ordered pair (n, m), and

$$\boldsymbol{\kappa}_\lambda = \boldsymbol{a}_n + \boldsymbol{k}_m, \quad \boldsymbol{A}_\lambda = \boldsymbol{\eta}_m \times \boldsymbol{B}_n. \tag{6.9}$$

Now evidently

$$\nabla \times \boldsymbol{A} + \alpha \boldsymbol{A} = \sum_\lambda (\mathrm{i}\boldsymbol{\kappa}_\lambda \times \boldsymbol{A}_\lambda + \alpha \boldsymbol{A}_\lambda) \, \mathrm{e}^{\mathrm{i}\boldsymbol{\kappa}_\lambda \cdot \boldsymbol{x}}, \tag{6.10}$$

and

$$\begin{aligned} \mathrm{i}\boldsymbol{\kappa}_\lambda \times \boldsymbol{A}_\lambda + \alpha \boldsymbol{A}_\lambda &= \mathrm{i}\boldsymbol{\kappa}_\lambda \times (\boldsymbol{\eta}_m \times \boldsymbol{B}_n) + \alpha \boldsymbol{\eta}_m \times \boldsymbol{B}_n \\ &= \mathrm{i}\boldsymbol{\kappa}_\lambda \times (\boldsymbol{\eta}_m \times \boldsymbol{B}_n) - \mathrm{i}\boldsymbol{\eta}_m \times (\boldsymbol{a}_n \times \boldsymbol{B}_n) \end{aligned}$$

using (6.2)

$$= \mathrm{i}[\boldsymbol{\eta}_m (\boldsymbol{k}_m \cdot \boldsymbol{B}_n) - \boldsymbol{a}_n (\boldsymbol{B}_n \cdot \boldsymbol{\eta}_m)]. \tag{6.11}$$

Hence from (6.7),

$$\begin{aligned} 2\delta^2 M &= \sum_\lambda [\boldsymbol{\kappa}_\lambda \times (\boldsymbol{\eta}_m^* \times \boldsymbol{B}_n^*)] \cdot [\boldsymbol{\eta}_m (\boldsymbol{k}_m \cdot \boldsymbol{B}_n) - \boldsymbol{a}_n (\boldsymbol{B}_n \cdot \boldsymbol{\eta}_m)] \\ &= \sum_{n,\, m} \{ |\boldsymbol{\eta}_m|^2 |\boldsymbol{k}_m \cdot \boldsymbol{B}_n|^2 - 4(\boldsymbol{a}_n \cdot \boldsymbol{\eta}_m^*) [\mathrm{Re}\, (\boldsymbol{B}_n \cdot \boldsymbol{k}_m) \boldsymbol{B}_n^*] \cdot \boldsymbol{\eta}_m \}. \end{aligned} \tag{6.12}$$

The second term sums to zero by virtue of the reality conditions (6.4), and we are left finally with

$$\delta^2 M = \tfrac{1}{2} \sum_{n,\, m} |\boldsymbol{\eta}_m|^2 |\boldsymbol{k}_m \cdot \boldsymbol{B}_n|^2; \tag{6.13}$$

since this expression is positive for all non-trivial $\boldsymbol{\eta}$, the stability is proved.†

This means that if the equilibrium (6.1) is disturbed in some way, then the fluid system will execute oscillations, which will be damped if viscous dissipation is present, about this equilibrium. Let us now analyse these oscillations on the assumption that

† A referee has pointed out that derivation of (6.12) from (6.7) is correct only if all the $\boldsymbol{\kappa}_\lambda$ are distinct. It is however not difficult to show that if $\boldsymbol{a}_n + \boldsymbol{k}_m = \boldsymbol{a}_n' + \boldsymbol{k}_m'$ (i.e. $\boldsymbol{\kappa}_\lambda = \boldsymbol{\kappa}_\lambda'$) then the corresponding contribution to $\delta^2 M$ is

$$\tfrac{1}{2} |\boldsymbol{\eta}_m (\boldsymbol{k}_m \cdot \boldsymbol{B}_n) + \boldsymbol{\eta}_m' (\boldsymbol{k}_m' \cdot \boldsymbol{B}_n')|^2$$

so that the conclusion following (6.13) is unaltered.

The referee has also pointed out that (6.12) in fact follows directly from (2.21) without the need to invoke the Beltrami property (6.2); this means that *all* space-periodic magnetostatic equilibria are stable to all disturbances of the form (6.5).

the scale L of the disturbance $\boldsymbol{\eta}(\boldsymbol{x}, t)$ (now regarded as a function of t as well as $\boldsymbol{x}$) is large compared with the scale α^{-1} of $\boldsymbol{B}^{\mathrm{E}}$. From (2.9), the perturbed magnetic field is

$$\boldsymbol{B} = \boldsymbol{B}^{\mathrm{E}} + \delta^1 \boldsymbol{B} + O(\eta^2), \tag{6.14}$$

and the associated Lorentz force may be written

$$(\boldsymbol{j} \times \boldsymbol{B})_i = \frac{\partial}{\partial x_j}(B_i B_j) - \frac{1}{2}\frac{\partial}{\partial x_i}\boldsymbol{B}^2. \tag{6.15}$$

The disturbance on the scale L will be driven by this Lorentz force, averaged over the 'inner' scale α^{-1}, i.e.

$$\langle \boldsymbol{j} \times \boldsymbol{B}\rangle_i = \frac{\partial}{\partial x_j}\langle B_i B_j\rangle - \frac{1}{2}\frac{\partial}{\partial x_i}\langle \boldsymbol{B}^2\rangle. \tag{6.16}$$

Now at leading order, $\boldsymbol{j}^{\mathrm{E}} \times \boldsymbol{B}^{\mathrm{E}} = 0$, so we need only concern ourselves with the term of order $\boldsymbol{\eta}$ in (6.16). The first variation of $\langle B_i B_j\rangle$ is

$$\delta^1\langle B_i B_j\rangle = \langle B_i^{\mathrm{E}} \delta^1 B_j\rangle + \langle B_j^{\mathrm{E}} \delta^1 B_i\rangle, \tag{6.17}$$

and

$$\langle B_i^{\mathrm{E}} \delta^1 B_j\rangle = \left\langle B_i^{\mathrm{E}}\left(B_k^{\mathrm{E}}\frac{\partial}{\partial x_k}\eta_j - \eta_k \frac{\partial}{\partial x_k} B_j^{\mathrm{E}}\right)\right\rangle = \langle B_i^{\mathrm{E}} B_k^{\mathrm{E}}\rangle \frac{\partial \eta_j}{\partial x_k} - \eta_k \left\langle B_i^{\mathrm{E}} \frac{\partial}{\partial x_k} B_j^{\mathrm{E}}\right\rangle, \tag{6.18}$$

since the average is over the scale α^{-1} on which $\boldsymbol{\eta}(\boldsymbol{x})$ is 'slowly varying'. Hence, since

$$\left\langle B_i^{\mathrm{E}}\frac{\partial}{\partial x_k} B_j^{\mathrm{E}}\right\rangle + \left\langle B_j^{\mathrm{E}}\frac{\partial}{\partial x_k} B_i^{\mathrm{E}}\right\rangle = \frac{\partial}{\partial x_k}\langle B_i^{\mathrm{E}} B_j^{\mathrm{E}}\rangle = 0,$$

if follows from (6.17) that

$$\delta^1\langle B_i B_j\rangle = C_{ijkl}\frac{\partial \eta_l}{\partial x_k}, \tag{6.19}$$

where

$$C_{ijkl} = \langle B_i^{\mathrm{E}} B_k^{\mathrm{E}}\rangle \delta_{jl} + \langle B_j^{\mathrm{E}} B_k^{\mathrm{E}}\rangle \delta_{il}. \tag{6.20}$$

Note that, from (6.1),

$$\langle B_i^{\mathrm{E}} B_k^{\mathrm{E}}\rangle = \frac{1}{2}\begin{bmatrix} B_2^2 + B_3^2 & \cdot & \cdot \\ \cdot & B_3^2 + B_1^2 & \cdot \\ \cdot & \cdot & B_1^2 + B_2^2 \end{bmatrix}. \tag{6.21}$$

The relationship (6.19) is a stress–strain relationship characteristic of an anisotropic elastic medium; the second term in (6.16) is accommodated through pressure variations in the incompressible fluid.

The 'isotropic' situation ($B_1 = B_2 = B_3$) is particularly simple. In this case (6.19) reduces to

$$\delta^1\langle B_i B_j\rangle = \tfrac{2}{3}M^{\mathrm{E}}\left(\frac{\partial \eta_j}{\partial x_i} + \frac{\partial \eta_i}{\partial x_j}\right), \tag{6.22}$$

where $M^{\mathrm{E}} = \frac{1}{2}\langle (\boldsymbol{B}^{\mathrm{E}})^2\rangle$. The associated contribution to (6.16) is

$$\frac{\partial}{\partial x_j}(\delta^1\langle B_i B_j\rangle) = \tfrac{2}{3}M^{\mathrm{E}}\nabla^2 \eta_i, \tag{6.23}$$

and the equation of motion of the fluid on the outer scale L is

$$\rho\frac{\partial^2 \boldsymbol{\eta}}{\partial t^2} = \tfrac{2}{3}M^{\mathrm{E}}\nabla^2\boldsymbol{\eta} + \text{viscous term}, \tag{6.24}$$

where ρ is the fluid density, assumed constant. The medium evidently supports waves which propagate with wave speed C_M given by

$$C_M = \left(\frac{2M^{\mathrm{E}}}{3\rho}\right)^{\frac{1}{2}}. \tag{6.25}$$

These waves are of course damped if due account is taken of viscosity.

We may note that the method described here may be readily adapted to describe oscillations on a large scale L of an arbitrary stable magnetostatic equilibrium $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$ with scale $l \ll L$. For example $\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x})$ might be a spatially complex field, statistically homogeneous and isotropic with respect to space averages (the end-product of the relaxation process described in M85, starting from an initial field $\boldsymbol{B}_0(\boldsymbol{x})$ with these same properties). Such oscillations are still evidently described by the wave equation (6.24).

7. Instability of Beltrami (ABC) flows

Consider now the Euler flow analogous to (6.1), viz.

$$\boldsymbol{u}^{\mathrm{E}} = (U_3 \cos \alpha z + U_2 \sin \alpha y,\ U_1 \cos \alpha x + U_3 \sin \alpha z,\ U_2 \cos \alpha y + U_1 \sin \alpha x). \tag{7.1}$$

The associated vorticity field is

$$\boldsymbol{\omega}^{\mathrm{E}} = -\alpha \boldsymbol{u}^{\mathrm{E}}. \tag{7.2}$$

This is the flow described as the ABC-flow by Dombré *et al.* (1986), after Arnol'd (1965*b*), Beltrami (1902) and Childress (1970). The flow, having maximal helicity, is a natural candidate for dynamo action, and has been studied in this context by Galloway & Frisch (1984) and Moffatt & Proctor (1985).

In order to consider the stability of the flow (7.1), we should first evaluate $\delta^2 K$, as given by (3.13). It will be sufficient to consider the particular displacement field

$$\boldsymbol{\eta} = \eta_0 (\cos kz, \sin kz, 0), \tag{7.3}$$

where k is a constant (positive or negative); this satisfies

$$\nabla \times \boldsymbol{\eta} = -k\boldsymbol{\eta}. \tag{7.4}$$

Using (7.2), and again replacing the integral by a space-average, (3.13) becomes

$$2\delta^2 K = \alpha^2[\langle \boldsymbol{A}^2\rangle - \langle (\nabla\phi)^2\rangle] + \alpha\langle \boldsymbol{A}\cdot\nabla\times\boldsymbol{A}\rangle, \tag{7.5}$$

where now $\boldsymbol{A} = \boldsymbol{\eta} \times \boldsymbol{u}^{\mathrm{E}}$ with components

$$\left.\begin{aligned} A_x &= \eta_0(U_2 \cos \alpha y + U_1 \sin \alpha x) \sin kz, \\ A_y &= -\eta_0(U_2 \cos \alpha y + U_1 \sin \alpha x) \cos kz, \\ A_z &= \eta_0[(U_1 \cos \alpha x + U_3 \sin \alpha z) \cos kz - (U_3 \cos \alpha z + U_2 \sin \alpha y) \sin kz], \end{aligned}\right\} \tag{7.6}$$

and ϕ is the space-periodic field satisfying

$$\nabla^2\phi = -\nabla\cdot\boldsymbol{A} = -\eta_0(\alpha - k)[U_1 \cos \alpha x \sin kz + U_2 \sin \alpha y \cos kz + U_3 \cos (\alpha - k) z], \tag{7.7}$$

i.e.

$$\phi = \frac{\eta_0(\alpha-k)}{\alpha^2+k^2}\{U_1 \cos \alpha x \sin kz + U_2 \sin \alpha y \cos kz\} + \frac{U_3\eta_0 \cos (\alpha-k) z}{\alpha - k}. \tag{7.8}$$

From these expressions, we may easily calculate

$$\langle A^2\rangle = \eta_0^2\,[\tfrac{3}{4}(U_1^2+U_2^2)+\tfrac{1}{2}U_3^2], \tag{7.9}$$

$$\langle(\nabla\phi)^2\rangle + \tfrac{1}{4}\eta_0^2\frac{(\alpha-k)^2}{\alpha^2+k^2}(U_1^2+U_2^2)+\tfrac{1}{2}\eta_0^2\,U_3^2, \tag{7.10}$$

and

$$\langle \boldsymbol{A}\cdot\nabla\times\boldsymbol{A}\rangle = -\tfrac{1}{2}\eta_0^2(k+\alpha)\,(U_1^2+U_2^2), \tag{7.11}$$

and hence, from (7.5), after simplification,

$$\delta^2 K = -\tfrac{1}{4}\eta_0^2\frac{(U_1^2+U_2^2)\,\alpha k^3}{\alpha^2+k^2}. \tag{7.12}$$

Since this expression changes sign as k changes sign, it is *indefinite* as regards sign, and so Arnol'd's (1966a) sufficient condition for stability is not satisfied. This does not necessarily mean that the flow is unstable, although the energy released by the perturbation, when $\alpha k > 0$, is available to augment the disturbance, and so instability may reasonably be anticipated in this case (following the clue provided by the Taylor–Couette situation discussed in §5 above).

To investigate this question further, let us adopt the 'mean-field' approach used in §6, i.e. suppose that the scale of $\boldsymbol{\eta}(\boldsymbol{x})$ is large compared with the scale of $\boldsymbol{u}^{\mathrm{E}}(\boldsymbol{x})$, i.e. that

$$k \ll \alpha. \tag{7.13}$$

The perturbed velocity is given by (3.6), i.e.

$$\boldsymbol{u} = \boldsymbol{u}^{\mathrm{E}} + (\boldsymbol{\eta}\times\boldsymbol{\omega}^{\mathrm{E}})_{\mathrm{s}} + O(\eta^2), \tag{7.14}$$

and we aim to calculate the Reynolds stress $\langle u_i\,u_j\rangle$ to order $\boldsymbol{\eta}$, the average now being over the scale α^{-1} To this order, we have

$$\langle u_i\,u_j\rangle = \langle u_i^{\mathrm{E}}\,u_j^{\mathrm{E}}\rangle + \left\langle u_i^{\mathrm{E}}\left[(\boldsymbol{\eta}\times\boldsymbol{\omega}^{\mathrm{E}})_j + \frac{\partial\phi'}{\partial x_j}\right]\right\rangle + (i\leftrightarrow j), \tag{7.15}$$

where

$$\nabla^2\phi' = -\nabla\cdot(\boldsymbol{\eta}\times\boldsymbol{\omega}^{\mathrm{E}}) = \alpha\nabla\cdot(\boldsymbol{\eta}\times\boldsymbol{u}^{\mathrm{E}}), \tag{7.16}$$

i.e. where, by comparison with (7.7),

$$\phi' = -\alpha\phi, \tag{7.17}$$

and $(i\leftrightarrow j)$ denotes repetition of the previous term with i and j interchanged.

Consider first the term

$$\langle u_i^{\mathrm{E}}(\boldsymbol{\eta}\times\boldsymbol{\omega}^{\mathrm{E}})_j\rangle = -\alpha\epsilon_{jkl}\langle u_i^{\mathrm{E}}\,u_l^{\mathrm{E}}\rangle\,\eta_k. \tag{7.18}$$

Evaluating the various components with $\boldsymbol{u}^{\mathrm{E}}$ and $\boldsymbol{\eta}$ given by (7.1) and (7.3) respectively, we find that the only non-zero terms are those for which

$$(i,j) = (1,3),\ (3,1),\ (2,3) \text{ or } (3,2), \tag{7.19}$$

and, for these,

$$\left.\begin{aligned} \langle u_1^{\mathrm{E}}(\boldsymbol{\eta}\times\boldsymbol{\omega}^{\mathrm{E}})_3\rangle + (1\leftrightarrow 3) &= \tfrac{1}{2}\alpha(U_3^2-U_1^2)\,\eta_0\sin kz,\\ \langle u_2^{\mathrm{E}}(\boldsymbol{\eta}\times\boldsymbol{\omega}^{\mathrm{E}})_3\rangle + (2\leftrightarrow 3) &= -\tfrac{1}{2}\alpha(U_3^2-U_2^2)\,\eta_0\cos kz.\end{aligned}\right\} \tag{7.20}$$

Consider now the term

$$\left\langle u_i^{\mathrm{E}}\frac{\partial\phi'}{\partial x_j}\right\rangle = -\alpha\left\langle u_i^{\mathrm{E}}\frac{\partial\phi}{\partial x_j}\right\rangle. \tag{7.21}$$

Again, using (7.1) and (7.8), we find that the only non-zero terms are the four given by (7.19); and for these

$$\left.\begin{aligned}\left\langle u_1^{\mathrm{E}}\frac{\partial\phi'}{\partial x_3}\right\rangle+(1\leftrightarrow 3)&=-\frac{\alpha}{2}\left(U_3^2-U_1^2\frac{\alpha^2-k^2}{\alpha^2+k^2}\right)\eta_0\sin kz,\\ \left\langle u_2^{\mathrm{E}}\frac{\partial\phi'}{\partial x_3}\right\rangle+(2\leftrightarrow 3)&=\frac{\alpha}{2}\left(U_3^2-U_2^2\frac{\alpha^2-k^2}{\alpha^2+k^2}\right)\eta_0\cos kz.\end{aligned}\right\}\tag{7.22}$$

Combining these results with (7.20), we find from (7.15)

$$\langle u_1 u_3\rangle=\frac{-\alpha k^2}{\alpha^2+k^2}U_1^2\eta_0\sin kz,$$

and

$$\langle u_2 u_3\rangle=\frac{\alpha k^2}{\alpha^2+k^2}U_2^2\eta_0\cos kz.\tag{7.23}$$

Now the effective force driving the large-scale perturbation is

$$\begin{aligned}-\frac{\partial}{\partial x_j}\langle u_i u_j\rangle&=-\left(\frac{\partial}{\partial z}\langle u_1 u_3\rangle,\frac{\partial}{\partial z}\langle u_2 u_3\rangle,0\right)\\&=\frac{\alpha k^3}{\alpha^2+k^2}(U_1^2\eta_0\cos kz,\ U_2^2\eta_0\sin kz,0)\end{aligned}$$

i.e.

$$-\frac{\partial}{\partial x_j}\langle u_i u_j\rangle=\frac{2\alpha k^3}{\alpha^2+k^2}K_{ij}^{\mathrm{E}}\eta_j,\tag{7.24}$$

where K_{ij}^{E} is the kinetic-energy matrix with components

$$(K_{ij}^{\mathrm{E}})=\frac{1}{2}\begin{bmatrix}U_1^2 & \cdot & \cdot\\ \cdot & U_2^2 & \cdot\\ \cdot & \cdot & U_3^2\end{bmatrix}\tag{7.25}$$

The factor $\alpha k^3(\alpha^2+k^2)^{-1}$ appearing in (7.24) is the same as that appearing in the expression (7.12) for δ^2K; thus, as expected, the force (7.24) which drives the large-scale perturbation is intimately related to the energy $(-\delta^2K)$ that is available to augment this perturbation. The equation of motion that is compatible with this description is

$$\rho\frac{\partial^2\eta_i}{\partial t^2}=\frac{2\alpha k^3}{\alpha^2+k^2}K_{ij}^{\mathrm{E}}\eta_j,\tag{7.26}$$

where $\boldsymbol{\eta}(\boldsymbol{x},t)$ is now regarded as a (slowly varying) function of $\boldsymbol{x}$ and t. Although this equation is not rigorously established by the above argument (which fails to take full account of the perturbation of the large-scale vorticity field by the small scale-velocity $\boldsymbol{u}^{\mathrm{E}}$), its structure is indicative of the instability that may be expected when $\delta^2K<0$.

We have carried out the above calculation for the particular displacement field (7.3). However the form of (7.26) now permits us to generalize the result to an arbitrary large-scale perturbation $\boldsymbol{\eta}(\boldsymbol{x},t)$. For, using (7.4), and expanding (7.26) in powers of $(k/\alpha)^2$, the equation may be written

$$\rho\frac{\partial^2\eta_i}{\partial t^2}=\frac{2}{\alpha}K_{ij}^{\mathrm{E}}\left(1+\frac{1}{\alpha^2}\nabla^2+\frac{1}{\alpha^4}\nabla^4+\ldots\right)\nabla^2(\boldsymbol{\nabla}\times\boldsymbol{\eta})_j,\tag{7.27}$$

a form that is presumably quite general.

The isotropic situation, in which

$$K_{ij}^{\mathrm{E}} = \tfrac{1}{3}K^{\mathrm{E}}\delta_{ij}, \tag{7.28}$$

is again particularly simple. If we retain only the leading term in (7.27), we then have

$$\rho\frac{\partial^2 \boldsymbol{\eta}}{\partial t^2} = \frac{2}{3\alpha}K^{\mathrm{E}}\nabla^2(\nabla\times\boldsymbol{\eta}), \tag{7.29}$$

which may be contrasted to (6.24). In this approximation, any helical mode for which

$$\nabla\times\boldsymbol{\eta} = -k\boldsymbol{\eta}, \quad \nabla\cdot\boldsymbol{\eta} = 0, \tag{7.30}$$

satisfies

$$\rho\frac{\partial^2 \boldsymbol{\eta}}{\partial t^2} = \frac{2}{3\alpha}K^{\mathrm{E}}k^3\boldsymbol{\eta}, \tag{7.31}$$

and is clearly unstable if $\alpha k > 0$, consistent with the remarks following (7.12).

It is apparent therefore that the flow (7.1) is unstable to large-scale helical disturbances having the same sign of helicity as the basic flow. This result is of great interest, since it indicates a mechanism for an 'inverse cascade' of helicity from large wavenumbers to small wavenumbers. There is every reason to believe that the same mechanism will be present for an *arbitrary* Euler flow (the analogue of the arbitrary magnetostatic equilibrium conceived in the final paragraph of §6). A similar point of view is adopted by Moiseev *et al.* (1984) in a discussion of turbulence with helicity, in which compressibility effects are regarded as important.

8. Conclusions

In this paper, we have discussed in general terms criteria for the stability of an arbitrary magnetostatic equilibrium and for the stability of the analogous Euler flow. A sufficient condition for stability of the magnetostatic equilibrium is

$$\delta^2 M > 0 \tag{8.1}$$

for all admissible displacement fields $\boldsymbol{\eta}(\boldsymbol{x})$, $\delta^2 M$ being defined by (2.14) or (2.21) (Bernstein *et al.* 1958). A sufficient condition for stability of the analogous Euler flow is given by Arnol'd's (1966*a*) condition

$$\delta^2 K > 0, \tag{8.2}$$

for all admissible $\boldsymbol{\eta}$, where $\delta^2 K$ is defined by (3.11) or (3.13). The alternative condition

$$\delta^2 K < 0 \tag{8.3}$$

for all admissible $\boldsymbol{\eta}$ has been shown by the explicit examples of §5 and §7 to be inapplicable, as recognized in the context of two-dimensional flows by Arnol'd (1966*b*). If $\delta^2 M$ and $\delta^2 K$ are evaluated for the same basic equilibrium field, we have shown (3.16) that, quite generally, the inequality

$$\delta^2 M + \delta^2 K > 0 \tag{8.4}$$

is satisfied.

The two-dimensional situation has been briefly discussed in §4. In this situation, there is a considerable simplification in that $\delta^2 K$ may be expressed explicitly in terms of $\boldsymbol{\eta}$ alone (and not involving the spatial gradient of $\boldsymbol{\eta}$). This same simplification occurs also for the cylindrically symmetric situation considered in §5. In both cases, this permits the extraction of a useful sufficient condition for the stability of the flow

((4.17) in the two-dimensional case, and (5.7) in the cylindrically symmetric case). The corresponding conditions for the magnetostatic problem are given by (4.14) and (5.5) respectively. It is noteworthy in the cylindrically symmetric problem that if the velocity component $v(r)$ (see (5.2)) is given by a power law

$$v(r) = Ar^{\lambda}, \tag{8.5}$$

then *both* the flow *and* the analogous magnetostatic equilibrium are stable provided

$$-1 < \lambda < 1. \tag{8.6}$$

For fully three-dimensional basic states, both $\delta^2 M$ and $\delta^2 K$ depend explicitly on $\nabla\boldsymbol{\eta}$ as well as $\boldsymbol{\eta}$. For the magnetostatic problem, a sufficient condition for stability may still be obtained on the basis of the inequality (2.29); but for the Euler-flow problem, the inequality (3.17) cannot be so exploited, since it still involves $\boldsymbol{\eta}$ in a non-trivial way.

The Beltrami field (6.1) (and analogously (7.1)) has been adopted as the prototype of a fully three-dimensional equilibrium. It has been shown in §6 that this magnetostatic equilibrium is stable to all disturbances which have a Fourier representation (6.5), and that, observed on a scale large compared with the scale α^{-1} of the equilibrium field, the medium exhibits an elastic stress–strain relationship (6.19). This means that waves of large wavelength can propagate through the 'lattice' provided by the field (6.1), the wave-speed being given (in the isotropic case) by (6.25).

The situation is very different as regards the stability of the Euler flow (7.1). Here $\delta^2 K$, evaluated for the particular disturbance (7.3), is given by (7.12), and is indefinite as regards sign. Hence conservation of energy places no constraint on the possible growth of frozen-field disturbances. A large-scale disturbance $\boldsymbol{\eta}(\boldsymbol{x}, t)$ generates a Reynolds stress $\langle u_i u_j \rangle$ whose components are given by (7.23), and the divergence of this Reynolds stress provides the force that excites the large-scale motion (an effect conveniently representated by equation 7.26). In the isotropic case, this disturbance equation takes the form (7.29), which suggests that the flow (7.1) is unstable to large-scale helical disturbances with the same sign of helicity as the basic flow. Since the factor $k^3(\alpha^2+k^2)^{-1}$ in (7.26) is a monotonic increasing function of k, the growth rate of the instabilities is stronger for larger k (within the limits permitted by the inequality (7.13)). This suggests that the presence of helicity in a flow such as (7.1), or more generally in a turbulent flow, may be conducive to the development of an inverse cascade towards progressively larger lengthscales. The Kelvin–Helmholtz type of instability, which is always associated with any vortex sheets that may be present in an Euler flow, must provide simultaneous energy transfer to *smaller* lengthscales. Which of the two mechanisms dominates in a turbulent flow will no doubt depend on the level of mean helicity in the energy-containing eddies of the turbulence.

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